Change of Variable for Multi-dimensional Integral

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Abstract

The change of variable theorem is proved under the sole hypothesis of differentiability of the transformation. Specifically, it is shown under this hypothesis that the transformed integral equals the given one over every measurable subset on which the transformation is injective; that countably many of these subsets cover the domain of invertibility; and that its complement – the domain of non-invertibility – is measurable and so may be broken off and handled separately.

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The change of variable formula in multi-dimensional calculus, $\int_{F(E)} \varphi\, dy = \int_E \varphi(F)|J_F|\, dz$, is usually established on the assumption that F is a one-to-one selfmap of n-space continuously differentiable with non-vanishing Jacobian J_F in an open domain containing E, $F(E)$ is a closed bounded domain and φ is real-valued integrable. The integrals make sense however – and thus the equality can be formulated – under more general conditions, the natural most general being just measurability of the domains and (Lebesgue) integrability of the integrands. Given the standard properties of the Lebesgue integral, it suffices to obtain the equality for the integrand $\varphi = 1$:

$$|F(E)| = \int_E \Delta F'\, dz,$$

a formula which will be shown valid on measurable subsets E on which F is injective. The absolute value of the Jacobian of F has been rewritten $\Delta F'$ (for reasons which will appear below).

If one wishes to cumulate these subsets (which will be shown to cover $\Delta F' > 0$ countably) one will need to replace the value of the measure on the left with the integral of the Banach indicatrix $N(y)$, which counts the number of elements of E sent by F on y. Also, the zero-set of $\Delta F'$ will be shown to have measure zero, which permits establishing the equality over it seperately.

This form has been attained gradually – see the successive editions of Rudin – and appears finally in his most recent edition as well as in Smith with, however, some superfluous restrictions and – what is perhaps more irritating – an appeal in both cases to Brouwer theory. A purely analytic proof can be extracted by specialization from a proof of the "area formula" in Evans *et al.* but as given there contains some gaps and unnecessary detours. Thus the proof below, accessible to anyone with a modest command of Lebesgue theory, should offer some appeal.

This paper was submitted to the Monthly in September 1998, thus was contemporaneous with Lax's which (although far less complete) appeared in the Monthly just when this one was rejected; a couple of years later, Lax published an extension (II in the References) which still does not tell the full story.

The sequel is organized as follows: The next five paragraphs motivate and introduce the "scale factor", which is the ratio by which a linear selfmap augments (or reduces) volume, and verifies its continuity for the operator norm. The following three paragraphs recall the definition of the derivative for vector-valued functions and show its Borel measurability as a function from the domain of differentibility to the operator normed linear selfmaps. The remaining three paragraphs then justify the integrability of the integrand, the break-off of its zero set, and the desired equality.

For real-valued functions of a single real variable, the derivative equals simultaneously two *a priori* distinct characteristics of a function F in the neighborhood of an argument x: it is the coefficient $F'(x)$ of the closest linear approximation to F near x and it is the slope of the tangent to the graph at x. These entities need to be distinguished in higher dimension; let's start with "slope."

For a linear function on the line, the absolute value of its slope is the

ratio of the length of the image of an interval by its length: it is the same for all
intervals and so may as well be defined as the length of the image of the unit semi-open interval $[0, 1)$.

For a linear selfmap L of n-space, define analogously the "slope" or "scale factor" ΔL as the ratio of the volume of the image of a box (i.e. a product of intervals) by its volume. This will be shown in a moment to be independent of the box and so may as well be defined as the volume of the image of the unit semi-open box (s.o.b.), i.e. the set of x's whose co-ordinates satisfy $0 \leq x_i < 1$. (In the plane the image of the unit square is the parallelogram spanned by the images of the co-ordinate vectors, whose cross product gives its area; to recognize ΔL in general as the absolute value of the determinant of the image vectors, one should arrange to obtain these as non–negative multiples of an orthonormal basis. However, this identification is not used in what follows.)

Divide the unit s.o.b. into disjoint equal size s.o.b.'s of side $\frac{1}{k}$: their volume is $\frac{1}{k^n}$ and there are k^n of them, sent into each other by translation. If L is invertible it sends them into disjoint parallelepipeds, also sent into each other by translation, hence of equal volume, whose union has volume ΔL – thus each image has volume $\left(\frac{1}{k^n}\right) \Delta L$. (If L is not invertible it maps into a lower dimensional space and all images have n-volume zero.) Any open U is a disjoint union of countably many s.o.b.'s, so its Lebesgue n-volume is also increased ΔL times by L and thus finally $|L(E)| = (\Delta L)|E|$ for every set E.

Δ is a continuous function on the L, topologized by uniform convergence on bounded sets – i.e. on any bounded with non-void interior. The image (by every L) of the closed unit box is compact, hence its epsilon neighborhoods are a base around the image: so a uniformly convergent sequence has image measures with lim sup $\leq$ the image measure of the limit L. To obtain the dual inequality, apply the same reasoning to the complement of the open unit box in some larger closed box.

A vector-valued function of a real variable can be differentiated in the classical way by taking the vector limit of the difference quotient, but this is already impossible even for a scalar-valued function of a vector variable. What one can do is to fix a vector u and take the limit of $\frac{F(z+hu)-F(z)}{h}$ as $h \downarrow 0$. This limit would then exist for, and be positive homogeneous in, every positive multiple pu in place of u. It thus suffices to postulate these limits for unit vectors u; they are called the directional derivatives (of F at z) in the directions u (the unit vectors are construed to point in the direction from the origin to their endpoint on the unit sphere; for a function of a single real variable this is the one-sided derivative and for co-ordinate u, the one-sided partials).

Assuming the directional derivatives exist in all directions u at z, one could define the "derivative" $F'(z)$ of F at z as the function which takes each u to the directional derivative $F'(z)u$. One will however require the approach of $\frac{F(z+hu)-F(z)}{h}$ to $F'(z)u$ to be uniform in the unit vectors u; and if $F'(z)$ is bounded this will result in $\frac{F(x)-F(z)}{|x-z|}$ bounded for x sufficiently close to z. This last may be shown to entail that $F'(z)pu := pF'(z)u,\ p \geq 0$, acts linearly on vectors pu for a.e. z: accordingly, we now

strengthen the definition by making $F'(z)$ a linear operator whose value at every v is the limit as $h \to 0$, uniform for bounded vector arguments v, of $\frac{F(z+hv)-F(z)}{h}$.

The derivative is thus a function from the domain, assumed measurable (follows from F continuous), of differentiability of F to the space of linear selfmaps of n-space topologized by uniform convergence on the unit ball[1] – a base at 0 is obtained by taking for every centered open ball the subset of those which send the unit ball into it. This function will next be shown to be (Borel) measurable: i.e., the inverse image of every open is measurable. Let $r_m \uparrow r$, the radius of centered open ball B in R^n, which is an increasing union of closed balls B_m of radius r_m with center zero. Then $F'(z)u \in B$ entails (by compactness of the u's) that all but finitely many (in k) of the continuous $k\left[F\left(z+\frac{u}{k}\right)-F(z)\right]$ belong to some B_m (recall) uniformly in u; the converse already holds if the belonging is uniform in some countable dense subset of the unit ball. Hence $\{z : F'(z)u \in B\}$ is the countable union (over m) of the sets where a countable intersection (over u) of lim sup (over k) of the modulus of continuous functions of z and u is $\le r_m$.

Since ΔL has been shown continuous for linear operators L, it follows that the composite, $\Delta F'(z)$, is a non-negative measurable function on a measurable subset E of n-space, hence has a well-defined Lebesgue integral $\int_E \Delta F'(z)dz$.

(It also follows that the set where $\Delta F'(z) = 0$ is measurable; since the equality for this set can be established by a simple calculation, we shall restrict to $\Delta F'(z) > 0$ in the sequel.)

If $F' := F'(z)$ is invertible at z then for $|v| <$ some $\delta, |F(z+v) - F(z) - F'v| \le \left[\varepsilon/\|F'^{-1}\|\right]|v| \le \varepsilon|F'v|$; thus $(1-\varepsilon)|F'x - F'z| \le |Fx - Fz| \le (1+\varepsilon)|F'x - F'z|$ for $|x-z| < \delta$. Observe that for fixed δ the set of z's satisfying (the extremes of) the first inequality is measurable: both terms are continuous in v and measurable in z so imposing it at a countable dense subset of the ball of radius δ reduces the implication to a countable set of inequalities between measurable functions. Since E is a countable union of these sets for a sequence of δ's, it suffices to establish the theorem on each of them. Then one can replace the fixed argument z of F' with any y in this subset and still obtain (for this δ) the extremes of the first, hence also the second, inequality with y in place of z. Restricting further to measurable subsets of diameter $< \delta$ on which the values of $|F'|$ lie between $(1-\varepsilon)$ and $(1+\varepsilon)$ of its value at z for every vector in the ball of radius δ, pass from the inequality with y in place of z back to F' at z, to attain $(1-\varepsilon)^2|F'x - F'y| \le |Fx - Fy| \le (1+\varepsilon)^2|F'x - F'y|$ even for $x, y \ne z$. (Since there is a countable cover by such sets, it suffices to restrict attention to each of them.) By the left inequality and invertibility of F' at z, F is one-to-one on this subset. From $|Fx - Fy| \le |Gx - Gy|$ for any one-to-one F and all x, y in some measurable set follows: FG^{-1} is non-expansive – Lip(1) – hence does not increase measure; so $|FE| \le |FG^{-1}GE| \le |GE|$ for each of its subsets E: thus $(1-\varepsilon)^{2n}\Delta F'|E| = (1-\varepsilon)^{2n}|F'(E)| \le |F(E)| \le (1+\varepsilon)^{2n}|F'(E)| =$

[1]This topology is defined by the norm $\|L\|$, the radius of the smallest ball centered at 0 which contains the image of the centered ball of radius one; by linearity this $\|L\|$ is also the Lipschitz constant.

$(1+\varepsilon)^{2n}\Delta F'|E|$. Since F is one-to-one, it preserves disjointness of subsets and since differentiable, measurability of subsets; hence $|F(\ )|$ is additive on measurable decompositions. By passing to the limit in the definition of the integral one is led at last to

$$(1-\varepsilon)^{2n}\int_E \Delta F'\,dz \leq |F(E)| \leq (1+\varepsilon)^{2n}\int_E \Delta F'\,dz$$

with arbitrary ε.

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